Deformation Theory

The combined notions of convergence, cotangent complex & infinitesimal cohesiveness are so important that one groups them together in the following:

Def'n: A prestack $\mathcal{X}$ is said to admit <u>deformation theory</u> if it satisfies:

(i) $\mathcal{X}$ is convergent, i.e. $\mathcal{X}(S) \xrightarrow{\sim} \lim\limits_{n \geq 0} \mathcal{X}({}^{\leq n}S)$;

(ii) $\mathcal{X}$ admits a cotangent complex, i.e. $T^*\mathcal{X} \in QCoh(\mathcal{X})^-$;

(iii) $\mathcal{X}$ is infinitesimally cohesive.

By the proposition from last time one have that assuming (i), (ii) + (iii) are equivalent to

$$\mathcal{X}(S_1' \sqcup_{S_1} S_2) \xrightarrow{\sim} \mathcal{X}(S_1') \times_{\mathcal{X}(S_1)} \mathcal{X}(S_2) \quad (\star) \quad \text{for any pushout}$$

$$\begin{tikzcd} S_1 \arrow[r] \arrow[d] & S_2 \arrow[d] \\ S_1' \arrow[r] & S_1' \sqcup_{S_1} S_2 \end{tikzcd}$$

where $S_1 \to S_1'$ is a nilpotent embedding.

Example: (i) Notice that the inclusion $Sch^{aff} \to Sch$ preserves pushouts by nilpotent embeddings.

Thus, any $Z \in Sch$ admits deformation theory.

Rk: Actually it is enough to check (★) for $S_1 \to S_1'$ a sq-zero extension by the Thm. from the last notes.

(ii) For $\{\mathcal{X}_i\}_I$ a filtered diagram of convergent prestacks. then if $\operatorname{colim}_I \mathcal{X}_i$ is convergent then it admits deformation theory.

In particular, ind-schemes admit deformation theory.

~~Before giving more consequences~~ Here are some consequences of admitting deformation theory.

Thm: Let $\mathcal{X}$ be a prestack admitting deformation theory. Suppose ${}^{cl}\mathcal{X}$ satisfy Zariski (resp. Nisnevich, étale) descent, then so does $\mathcal{X}$.

Idea: Consider.

$$\begin{tikzcd} S_0' \arrow[r, "\text{étale}"] \arrow[d] & S_0 \arrow[d] \\ S_1' \arrow[r, "\text{étale}"'] & S_1 \end{tikzcd}$$

& ${}^{cl}S_1' = S_0'$

${}^{cl}S_1 = S_0$.

Enough to check (Δ) is an isomorphism:

$$\begin{tikzcd} \mathcal{X}(S_0) \arrow[r, "\simeq"] & \lim_{\Delta^{op}} \mathcal{X}((S_0'/S_0)^\bullet) \\ \mathcal{X}(S_1) \arrow[u] \arrow[r] & \lim_{\Delta^{op}} \mathcal{X}((S_1'/S_1)^\bullet) \arrow[u, "\alpha"'] \\ \mathrm{Fib}(\mathcal{X}(S_1) \to \mathcal{X}(S_0)) \arrow[u] \arrow[r, "(\Delta)"'] & \mathrm{Fib}(\alpha) \arrow[u] \end{tikzcd}$$

$\mathrm{Fib}(\alpha) \simeq \mathrm{Tot}(\text{null-homotopies } T^*_x\mathcal{X} \to \mathcal{F}^\bullet)$

But: $\mathrm{Fib}(\mathcal{X}(S_1) \to \mathcal{X}(S_0))$ is null-homotopies of $T^*_x\mathcal{X} \to \mathcal{F}$

$x: S_0 \to \mathcal{X}$, for some $\mathcal{F} \in QCoh(S_0)^{\geq 0}$.

And similarly: $\mathcal{F}^\bullet := h^\bullet(\mathcal{F})$

$h^\bullet: S_0'^\bullet \to S_0$.

The point is that in the Totalization one can actually pass to a finite indexing set, so $\mathrm{Hom}(T^*_x\mathcal{X}, -)$ commutes w/ finite limits. ■

Here is another very useful consequence:

<u>Thm:</u> Let $f: \mathcal{X} \to \mathcal{Y}$ be a map of prestacks that admit deformation theory, and suppose that for every $x: S_0 \to \mathcal{X}$, $S_0 \in {}^{cl}\mathrm{Sch}^{\mathrm{aff}}$, one has:

$$T^*_y\mathcal{Y} \xrightarrow{\simeq} T^*_x\mathcal{X} \qquad y = f \circ x.$$

and that ${}^{cl}f: {}^{cl}\mathcal{X} \xrightarrow{\simeq} {}^{cl}\mathcal{Y}$ is an isomorphism.

Then f is an isomorphism.

<u>Pf:</u> Enough by induction on sq.-zero extensions to prove that for any $S \hookrightarrow S'$ a sq. zero. extension one has:

$$\mathrm{Maps}_{S/}(S', \mathcal{X}) \simeq \mathrm{Maps}_{S/}(S', \mathcal{Y}).$$

is

~~$\mathrm{Hom}(T^*_x\mathcal{X}, \mathcal{F})$ for see~~ is. <u>for see $\mathcal{F} \in \mathrm{QCoh}(S)^{\leq 0}$.</u>

$$\left\{\begin{tikzcd} T^*_x\mathcal{X} \arrow[r] & \mathcal{F} \arrow[d, "\mathrm{id}_\mathcal{F}"] \\ 0 \arrow[r] & \mathcal{F} \end{tikzcd}\right\} \simeq \left\{\begin{tikzcd} T^*_y\mathcal{X} \arrow[r] & \mathcal{F} \arrow[d, "\mathrm{id}_\mathcal{F}"] \\ 0 \arrow[r] & \mathcal{F} \end{tikzcd}\right\}.$$

■

Finally the next result gives a criterion for checking the l.a.f.t. condition

<u>Thm:</u> Let $\mathcal{X} \in \mathrm{PStk}_{\mathrm{def}}$, if

(i) ${}^{cl}\mathcal{X} \in {}^{cl}\mathrm{PStk}_{\mathrm{lft}}$, i.e. ${}^{cl}\mathcal{X}(\mathrm{colim}_I\, T_i) \simeq \lim_{I^{op}} {}^{cl}\mathcal{X}(T_i)$

where $T_i \in {}^{cl}\mathrm{Sch}^{\mathrm{aff}}$.

(ii) $\forall\, T \in {}^{cl}\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ $x: T \to \mathcal{X}$ one has
$T^*_x\mathcal{X} \in \mathrm{Pro}(\mathrm{Coh}(T))$ ~~(little Pro, it should be $\mathrm{Pro}(\mathrm{QCoh}(X))_{\mathrm{laft}}$.)~~

Rk: (The def'n of $\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X}))_{\mathrm{laft}}$ is:
- Φ is convergent.
- $\forall m \geq 0$ $\Phi|_{\mathrm{QCoh}(\mathcal{X})^{\geq -m, \leq 0}} : \mathrm{QCoh}(\mathcal{X})^{\geq -m, \leq 0} \to \mathrm{Spc}$ commutes w/ filt. colimits.)

For $\mathcal{X} \in \mathrm{Sch}_{\mathrm{aft}}$, $\mathrm{Pro}(\mathrm{QCoh}(\mathcal{X}))_{\mathrm{laft}} = \mathrm{Pro}\,\mathrm{Coh}(\mathcal{X})$.

We notice that the condition on $T_x^* \mathcal{X}$ can be check in cohomology, i.e.

$$H^i(T_x^* \mathcal{X}) \in \mathrm{Pro}(\mathrm{Coh}(T)^{\heartsuit}) \quad \forall\, i \leq 0.$$

Moreover, if T is eventually coconnective, i.e. $T \in {}^{\leq n}\mathrm{Sch}^{\mathrm{aff}}$ for some n, the $\mathrm{Perf}(T) \subset \mathrm{Coh}(T)$, so

$$H^i(T_x^* \mathcal{X}) \in \mathrm{Perf}(T)^{\heartsuit} \quad \forall\, i \leq 0 \Rightarrow \text{our condition (ii) above.}$$

~~Q: When can we determine that $T_x^* \mathcal{X} \in \mathrm{Pro}(\mathrm{QCoh}(T))$ for all $x : T \to \mathcal{X}$, $T \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ from $T_y^* \mathcal{X}$ for $y : S \to \mathcal{X}$ w/ $S \in$~~

The proof of the above theorem is a bit more involved we refer the reader to [GR-II, Chapter 1, §9].