

Deformation Theory

The combined notions of convergence, tangent complex & infinitesimal cohesiveness are so important that one groups them together in the following:

Def'n: A prestack \mathcal{X} is said to admit deformation theory if it satisfies:

(i) \mathcal{X} is convergent, i.e. $\mathcal{X}(S) \xrightarrow{\cong} \lim_{n \geq 0} \mathcal{X}(S_n S)$;

(ii) \mathcal{X} admits a tangent complex, i.e. $T^* \mathcal{X} \in \mathcal{QGrh}(\mathcal{X})^-$;

(iii) \mathcal{X} is infinitesimally cohesive.

By the proposition from last time one has that assuming (i), (ii) + (iii) are equivalent to

$$\mathcal{X}(S_1' \amalg_{S_1} S_2) \xrightarrow{\cong} \mathcal{X}(S_1') \times_{\mathcal{X}(S_1)} \mathcal{X}(S_2) \quad (*) \quad \text{for any pushout}$$

$$S_1 \rightarrow S_2$$

$$\begin{array}{ccc} \downarrow & \downarrow & \\ S_1' & \rightarrow & S_1' \amalg_{S_1} S_2 \end{array}$$

where $S_1 \rightarrow S_1'$ is a nilpotent embedding.

Example: (i) Notice that the inclusion $Sch^{aff} \hookrightarrow Sch$ preserves pushouts by nilpotent embeddings.

Thus, any $Z \in Sch$ admits deformation theory.

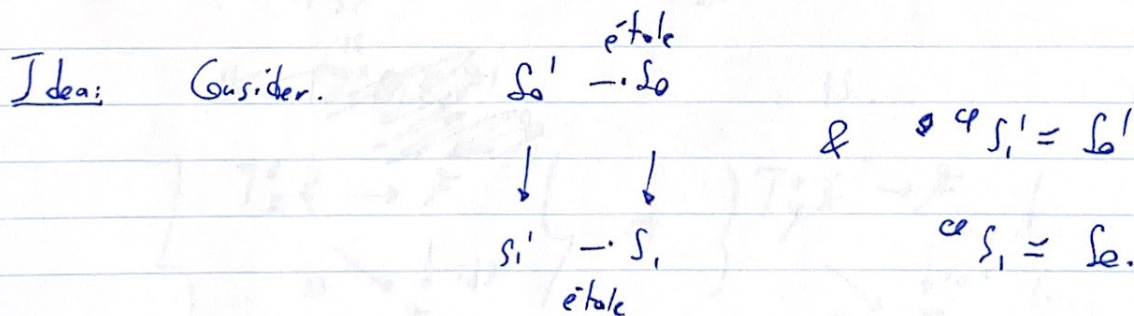
Rk: Actually it is enough to check (\star) for $S_1 \rightarrow S_1'$ a sq-zero extension, by the Thm. from the last notes.

(ii) For $\{X_i\}_{i \in I}$ a filtered diagram of convergent prestacks. Then if $\varinjlim X_i$ is convergent then it admits deformation theory.

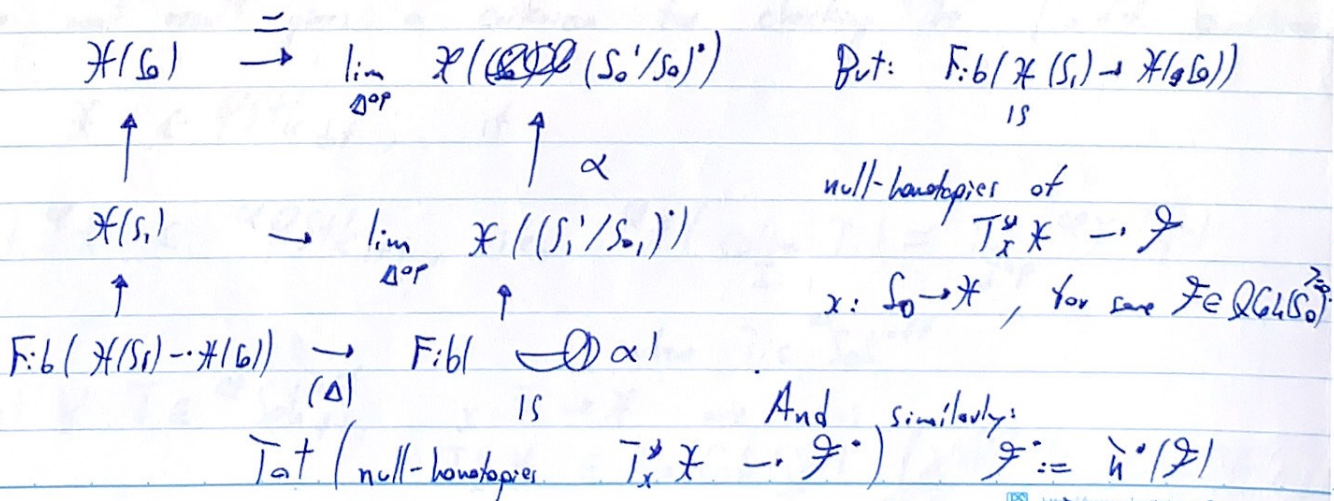
In particular, ind-schemes admit deformation theory.

~~Before giving more consequences~~ Here are some consequences of admitting deformation theory.

Thm: Let X be a prestack admitting deformation theory. Suppose X satisfy Zariski (resp. Nisnevich, étale) descent, then so does X .



Enough to check (Δ) is an isomorphism:



The point is that in the Totalization one can actually pass to a finite indexing set, so $\text{Hom}(T_x^* X, -)$ commutes w/ finite limits.

Here is another very useful consequence:

Thm: Let $f: X \rightarrow Y$ be a map of presheaves that admit deformation theory, and suppose that for every $x: S_0 \rightarrow X$, $S_0 \in \text{Sch}^{\text{aff}}$, one has:

~~$T_y^* Y \cong T_x^* X$~~ $T_y^* Y \cong T_x^* X$ $y = f \circ x$

and that $\text{df}_x: \mathcal{O}_x \cong \mathcal{O}_y$ is an isomorphism.

Then f is an isomorphism.

Pf: Enough by induction on sq-zero extensions to prove that for any $S \subset S'$ a sq-zero extension one has:

$$\text{Maps}_{S'}(S, X) \cong \text{Maps}_{S'}(S, Y)$$

~~$\text{Hom}(T_x^* X, \mathcal{O}_x)$~~ is. for $S \in \text{Coh}(S)^{\text{co}}$

$$\left\{ \begin{array}{c} T_x^* X \rightarrow \mathcal{O}_x \\ \downarrow \text{id} \\ \mathcal{O}_x \end{array} \right\} \cong \left\{ \begin{array}{c} T_y^* X \rightarrow \mathcal{O}_y \\ \downarrow \text{id} \\ \mathcal{O}_y \end{array} \right\}$$

Finally the next result gives a criterion for checking the l.a.f.t. condition

Thm: Let $X \in \text{PStk}_{\text{def}}$, if

(i) $\mathcal{O}_X \in \text{PStk}_{\text{lft}}$, i.e. $\mathcal{O}_X(\text{colim}_I T_i) \cong \lim_{\text{top}} \mathcal{O}_X(T_i)$

where $T_i \in \text{Sch}^{\text{aff}}$.

(ii) $\forall T \in \text{PStk}_{\text{lft}}$

$x: T \rightarrow X$ one has $T_x^* X \in \text{Pro}(\text{Coh}(T))$

~~little bit, it should be $\text{Pro}(\text{Coh}(X))_{\text{lft}}$.~~

Rk: The defn of $\text{Pro}(\text{Coh}(X))_{\text{latt}}$ is:

- Φ is concept.

- $\forall m \geq 0$ $\Phi|_{\text{Coh}(X)^{\geq -m, \leq 0}} : \text{Coh}(X)^{\geq -m, \leq 0} \rightarrow \text{Spec}$ commutes w/ f.l.t. colimits.

For $X \in \text{Sch}_{\text{aff}}$, $\text{Pro}(\text{Coh}(X))_{\text{latt}} = \text{Pro}(\text{Coh}(X))$.

We notice that the condition on $T_x^* X$ can be check in cohomology, i.e.

$$H^i(T_x^* X) \in \text{Pro}(\text{Coh}(T)^{\heartsuit}) \quad \forall i \leq 0.$$

Moreover, if T is centrally connected, i.e. $T \in \text{Sch}_{\text{aff}}$ for some n , $\text{Perf}(T) \in \text{Coh}(T)$, so.

$$H^i(T_x^* X) \in \text{Perf}(T)^{\heartsuit} \quad \forall i \leq 0 \Rightarrow \text{our condition. } \textcircled{ii} \text{ a base.}$$

~~Q: When can we determine that $T_x^* X \in \text{Pro}(\text{Coh}(T))$ for all $T \in \text{Sch}_{\text{aff}}$ from $T_x^* X$ for $\forall X \in \text{Sch}_{\text{aff}}$ w/ $X \in \text{Sch}_{\text{aff}}$?~~

The proof of the above theorem is a bit more involved we refer the reader to [GR-II, Chapter 1, § 9].